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Computation of Partial Derivatives for Ground Tracking of Space Shuttle Orbits

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Ground radar tracking data has been processed for many years to determine the trajectories of manned space vehicles. Recently, it was decided that venting and atmospheric drag, as well as Space Shuttle vehicle and payload orbits, were to be determined in real time from the tracking data. This has required a modification and extension of the method used for many years to calculate partial derivatives of the position and velocity components of space vehicles. The way in which this was done is described in detail, but also placed in a general context which might prove useful for other important applications.

Introduction

DETERMINATION of the trajectories of Space Shuttle vehicles and their payloads, using continually updated radar tracking data, is an operational requirement of the ground system at the NASA Johnson Space Center. The requirement is necessary to track, predict, and control the trajectories of both manned and unmanned spacecraft. The computer programs that will perform this function in the ground control installation are modifications and extensions of techniques which have been used for several years to determine the trajectories of manned vehicles from radar tracking data.

Such tracking has required the determination of state partial derivatives (sensitivity coefficients) of the position and velocity of a vehicle at many different times along a trajectory with respect to vehicle position and velocity at an initial reference time. A method developed by the present author[1] has been used to do this for the orbital portion of the vehicle trajectories for many years. The method uses analytical formulas for the state partial derivatives of unperturbed, two-body motion to approximate the corresponding partial derivatives for the perturbed motion.

Recently, it was decided to estimate parameters which describe effects such as atmospheric drag and vehicle venting in real time from radar tracking data. These parameters thus have to be estimated along with the initial reference position and velocity that determine the vehicle trajectory. This, in turn, requires the determination of the parameter partial derivatives of the vehicle position and velocity at different measurement times with respect to the parameters, as well as the state partial derivatives with respect to the initial position and velocity.

A quadrature formula[2] was applied to calculate these additional partial derivatives with respect to the required parameters. Also, chain computation was used for both state and parameter partials. The result was shown to offer advantages of speed, accuracy, simplicity, and compactness for the computation of the additional derivatives. Therefore, the approach was incorporated into the ground computer programs that will be used at the Johnson Space Center to track both the Space Shuttle vehicle and its payloads.

The first three sections below summarize general results concerning first-order variations of the solution of a system of simultaneous, ordinary differential equations. A later section outlines the actual method used to compute all the required partial derivatives in the ground control installation at the NASA Johnson Space Center. The example should serve to illustrate how the general results in the first three sections can be used in other important practical applications.

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Background and Notation

Let a system of simultaneous ordinary differential equations be written in the first-order form

$$\mathrm{d}s/\mathrm{d}t=f \tag{1}$$

where t is the (scalar) independent variable. The n elements of the n-dimensional state vector s are the dependent variables of the system Eq. (1) of n simultaneous scalar first-order, ordinary differential equations. The n-dimensional vector function

$$f=f(s,c,t) \tag{2}$$

is in general an explicit function of s, c, and t. The m elements of the m-dimensional vector c are parameters for which

$$\mathrm{d}c/\mathrm{d}t=0 \tag{3}$$

so that c does not vary with t.

Standard matrix notation is assumed with a vector represented by a single-column matrix. The usual definitions of matrix addition, multiplication, transposition, and inversion hold. The notation 0 denotes a zero matrix, while I denotes a (square) unit matrix. The notation A^T denotes the transpose of a matrix A, while B^{-1} denotes the matrix inverse of a (square nonsingular) matrix B. The application of any scalar operator to a matrix means the matrix formed by applying the scalar operator to each element of the matrix. Thus, $\mathrm{d}s/\mathrm{d}t$ in Eq. (1) denotes the vector of total derivatives of elements of s with respect to t.

Initial conditions for Eqs. (1) and (3) will be denoted by

$$t=t_0 \Rightarrow s=s_0 \tag{4}$$

$$t=t_0 \Rightarrow c=c_0 \tag{5}$$

where s_0, c_0, t_0 are given finite initial values of s, c, t. Although the solution

$$c=c_0 \quad \text{for all} \quad t \tag{6}$$

of Eq. (3) with Eq. (5) is trivial, so that c and c_0 are always the same as t varies, it is useful to distinguish c at t formally from its initial value c_0 at t_0 in Eq. (5). More generally, subscripts on t denote particular values of t, and the same subscript on other variables indicates their value for the corresponding value of t. For example,

$$f_0 = f(s_0, c_0, t_0) = f(s_0, c, t_0) \tag{7}$$

is the initial value of f in Eq. (2).

It is assumed that the function f of s,c,t in Eq. (2) satisfies conditions that ensure the existence and uniqueness of the solution

$$s = \sigma(s_0, c_0, t_0, t) \tag{8}$$

of Eq. (1) as a function σ of s_0, c_0, t_0, and t for values of t in some interval of interest about t_0. The notation σ in Eq. (8) denotes the solution function so defined, while s denotes its value at t. It is also assumed that the inverse function

$$s_0 = \sigma(s, c, t, t_0) \tag{9}$$

to Eq. (8) is defined for values of t in the same interval about t_0. The notation σ is used in both Eqs. (8) and (9) to denote the same function with different arguments. The particular case

$$s_0 = \sigma(s_0, c_0, t_0, t_0) \tag{10}$$

of Eqs. (8) or (9) for $t = t_0$ is a more formal way of writing the initial condition Eq. (4).

It is also assumed that the partial derivatives

$$\partial f/\partial s \equiv \partial f(s,c,t)/\partial s \tag{11}$$

$$\partial f/\partial c \equiv \partial f(s,c,t)/\partial c \tag{12}$$

$$\partial f/\partial t \equiv \partial f(s,c,t)/\partial t \tag{13}$$

of f in Eq. (2) exist in the interval of interest in t. The notation $\partial f/\partial t$ in Eq. (13) denotes the column matrix of partial derivatives of each of the n elements of the vector function $f(s,c,t)$ with respect to t. The notation $\partial f/\partial c$ in Eq. (12) denotes the $n \times m$ matrix of partial derivatives of the n elements of f with respect to the m elements of c. The element in row i and column j of the matrix $\partial f/\partial c$ is the partial derivative of element i of f with respect to element j of c. The notation $\partial f/\partial s$ in Eq. (11) accordingly denotes the (square) $n \times n$ matrix of partial derivatives of the n elements of f with respect to the n elements of s. From the chain rule for total differentiation,

$$\mathrm{d}f/\mathrm{d}t = [\partial f/\partial s]f + [\partial f/\partial t] \tag{14}$$

is the total derivative of $f(s,c,t)$ in Eq. (2) with respect to t.

Partial Derivatives and Variations

It is also assumed that f in Eq. (2), and its partials in Eqs. (11-13), satisfy conditions that ensure the existence and uniqueness of the partials

$$\partial s/\partial s_0 \equiv \partial\sigma(s_0, c_0, t_0, t)/\partial s_0 \tag{15}$$

$$\partial s/\partial c_0 \equiv \partial\sigma(s_0, c_0, t_0, t)/\partial c_0 \tag{16}$$

$$\partial\sigma(s_0, c_0, t_0, t)/\partial t_0 = -[\partial s/\partial s_0]f_0 \tag{17}$$

$$\partial\sigma(s_0, c_0, t_0, t)/\partial t = f \tag{18}$$

of Eq. (8) and

$$\partial s_0/\partial s \equiv \partial\sigma(s, c, t, t_0)/\partial s \tag{19}$$

$$\partial s_0/\partial c \equiv \partial\sigma(s, c, t, t_0)/\partial c \tag{20}$$

$$\partial\sigma(s, c, t, t_0)/\partial t = -[\partial s_0/\partial s]f \tag{21}$$

$$\partial\sigma(s, c, t, t_0)/\partial t_0 = f_0 \tag{22}$$

of Eq. (9) for values of t in the interval of interest about t_0. The relations (15), (16) and (19), (20) are merely convenient notational definitions. The relation (17) and its counterpart Eq. (21), are true but not obvious. The relation (18) is merely a more formal way of writing the differential equations (1) since s_0, c_0, t_0, as well as t, are treated as variables in Eq. (18). A similar interpretation holds for Eq. (22).

In the notation of the calculus of variations,

$$\delta s = [\partial s/\partial s_0][\delta s_0 - f_0 \delta t_0] + [\partial s/\partial c_0]\delta c_0 + f\delta t \tag{23}$$

is the total first-order variation δs in s in terms of the first-order variations δs_0 in s_0, δt_0 in t_0, δc_0 in c_0, and δt in t. The inverse relation

$$\delta s_0 = [\partial s_0/\partial s][\delta s - f\delta t] + [\partial s_0/\partial c]\delta c + f_0 \delta t_0 \tag{24}$$

expresses the total first-order variation δs_0 in s_0, due to the first-order variations δs in s, δt in t, δc in c, and δt_0 in t_0. Also

$$\delta c = \delta c_0 \tag{25}$$

follows trivially from Eq. (6). But the variations δc_0 are independent of $\delta s_0, \delta t_0, \delta t$, while the variations δc are independent of δs, δt, δt_0.

It will be assumed that $\partial s/\partial s_0$ and $\partial s/\partial c_0$ in Eqs. (15) and (16), as well as $\partial s_0/\partial s$ and $\partial s_0/\partial c$ in Eqs. (19) and (20), are defined as continuous matrix functions for values of t in the interval of interest about t_0. But finite discontinuities in f in Eq. (2), or its partials in Eqs. (11-13) at particular values of t including t_0, can occur. Then the formulas Eqs. (23) and (24) have to be interpreted carefully, because the value of f can depend on the sign of δt, or the value of f_0 can depend on the sign of δt_0.

The state partials $\partial s/\partial s_0$ (partial derivatives of s with respect to the initial state s_0) in Eq. (15) satisfy the state variational equations and initial conditions

$$\mathrm{d}[\partial s/\partial s_0]/\mathrm{d}t = [\partial f/\partial s][\partial s/\partial s_0] \tag{26}$$

$$t = t_0 \Rightarrow \partial s/\partial s_0 = I \tag{27}$$

while the parameter variational equations and initial conditions

$$\mathrm{d}[\partial s/\partial c_0]/\mathrm{d}t = [\partial f/\partial s][\partial s/\partial c_0] + [\partial f/\partial c] \tag{28}$$

$$t = t_0 \Rightarrow \partial s/\partial c_0 = 0 \tag{29}$$

hold for the parameter partials $\partial s/\partial c_0$ (partial derivatives of s with respect to the initial parameters c_0) in Eq. (16).

The inverse state partials $\partial s_0/\partial s$ (partial derivatives of s_0 with respect to the state s) in Eq. (19) satisfy the adjoint state variational equations and initial conditions

$$\mathrm{d}[\partial s_0/\partial s]/\mathrm{d}t = -[\partial s_0/\partial s][\partial f/\partial s] \tag{30}$$

$$t = t_0 \Rightarrow \partial s_0/\partial s = I \tag{31}$$

while the adjoint parameter variational equations and initial conditions

$$\mathrm{d}[\partial s_0/\partial c]/\mathrm{d}t = -[\partial s_0/\partial s][\partial f/\partial c] \tag{32}$$

$$t = t_0 \Rightarrow \partial s_0/\partial c = 0 \tag{33}$$

hold for the inverse parameter partials $\partial s_0/\partial c$ (partial derivatives of s_0 with respect to the parameters c) in Eq. (20).

The partials $\partial s/\partial s_0$ and $\partial s_0/\partial s$ satisfy the equivalent relations

$$[\partial s_0/\partial s][\partial s/\partial s_0]=I \tag{34}$$

$$\Leftrightarrow [\partial s/\partial s_0][\partial s_0/\partial s]=I \tag{35}$$

$$\Leftrightarrow [\partial s_0/\partial s]=[\partial s/\partial s_0]^{-1} \tag{36}$$

$$\Leftrightarrow [\partial s/\partial s_0]=[\partial s_0/\partial s]^{-1} \tag{37}$$

while the equivalent relations

$$[\partial s_0/\partial c]=-[\partial s_0/\partial s][\partial s/\partial c_0] \tag{38}$$

$$\Leftrightarrow [\partial s/\partial c_0]=-[\partial s/\partial s_0][\partial s_0/\partial c] \tag{39}$$

hold for the parameter partials $\partial s/\partial c_0$ and $\partial s_0/\partial c$. The relations (36-39) can often be used to compute some of these partials from the others. For example, if $\partial s/\partial s_0$ and $\partial s/\partial c_0$ have been determined by integrating Eq. (26) with Eq. (27) and Eq. (28) with Eq. (29), then Eq. (36) gives $\partial s_0/\partial s$, and Eq. (38) gives $\partial s_0/\partial c$.

In the particular case

$$f=f(s,c) \tag{40}$$

of Eq. (2), for which f does not depend explicitly on t, the system of differential equations (1) is said to be autonomous. Then obviously

$$f_0=f(s_0,c_0)=f(s_0,c) \tag{41}$$

replaces Eq. (7). The function Eq. (8) can then be written

$$s=\sigma(s_0,c_0,t-t_0) \tag{42}$$

since the solution s is a function of the interval $(t-t_0)$. Also

$$s_0=\sigma(s,c,t_0-t) \tag{43}$$

replaces Eq. (9) and

$$s_0=\sigma(s_0,c_0,0) \tag{44}$$

replaces Eq. (10), while

$$\partial f/\partial t\equiv 0 \tag{45}$$

replaces Eq. (13) so that

$$\mathrm{d}f/\mathrm{d}t=[\partial f/\partial s]f \tag{46}$$

replaces Eq. (14).

In addition, the equivalent relations

$$f=[\partial s/\partial s_0]f_0 \tag{47}$$

$$\Leftrightarrow f_0=[\partial s_0/\partial s]f \tag{48}$$

hold for autonomous equations (1) with Eq. (40), so that

$$\delta s=[\partial s/\partial s_0]\delta s_0+[\partial s/\partial c_0]\delta c_0+f\delta(t-t_0) \tag{49}$$

replaces Eq. (23) and

$$\delta s_0=[\partial s_0/\partial s]\delta s+[\partial s_0/\partial c]\delta c+f_0\delta(t_0-t) \tag{50}$$

replaces Eq. (24).

Integral Formulas and Chain Rules

Let t_1 be a particular value of t for which $t_1\neq t_0$ in general. Then

$$s_1=\sigma(s_0,c_0,t_0,t_1) \tag{51}$$

$$=s_0+\int_{t_0}^{t_1} f(s,c,t)\,\mathrm{d}t \tag{52}$$

is the value s_1 of s at $t=t_1$ and

$$c_1=c_0 \tag{53}$$

is the value of c at $t=t_1$. Also, the inverse relations

$$s_0=\sigma(s_1,c_1,t_1,t_0) \tag{54}$$

$$=s_1-\int_{t_0}^{t_1} f(s,c,t)\,\mathrm{d}t \tag{55}$$

to Eqs. (51) and (52) hold.

The state partials of Eq. (51), with respect to s_0, satisfy the relations

$$\partial s_1/\partial s_0\equiv\partial\sigma(s_0,c_0,t_0,t_1)/\partial s_0 \tag{56}$$

$$=[\partial s_0/\partial s_1]^{-1} \tag{57}$$

$$=I+\int_{t_0}^{t_1}[\partial f/\partial s][\partial s/\partial s_0]\,\mathrm{d}t \tag{58}$$

$$=I+\int_{t_0}^{t_1}[\partial s_1/\partial s][\partial f/\partial s]\,\mathrm{d}t \tag{59}$$

while

$$\partial s_1/\partial c_0\equiv\partial\sigma(s_0,c_0,t_0,t_1)/\partial c_0 \tag{60}$$

$$=-[\partial s_1/\partial s_0][\partial s_0/\partial c_1] \tag{61}$$

$$=\int_{t_0}^{t_1}([\partial f/\partial s][\partial s/\partial c_0]+[\partial f/\partial c])\,\mathrm{d}t \tag{62}$$

$$=\int_{t_0}^{t_1}[\partial s_1/\partial s][\partial f/\partial c]\,\mathrm{d}t \tag{63}$$

hold for the parameter partials of Eq. (51), with respect to c_0.

The inverse state partials of Eq. (54), with respect to s_1, satisfy the relations

$$\partial s_0/\partial s_1\equiv\partial\sigma(s_1,c_1,t_1,t_0)/\partial s_1 \tag{64}$$

$$=[\partial s_1/\partial s_0]^{-1} \tag{65}$$

$$=I-\int_{t_0}^{t_1}[\partial f/\partial s][\partial s/\partial s_1]\,\mathrm{d}t \tag{66}$$

$$=I-\int_{t_0}^{t_1}[\partial s_0/\partial s][\partial f/\partial s]\,\mathrm{d}t \tag{67}$$

while

$$\partial s_0/\partial c_1\equiv\partial\sigma(s_1,c_1,t_1,t_0)/\partial c_1 \tag{68}$$

$$=-[\partial s_0/\partial s_1][\partial s_1/\partial c_0] \tag{69}$$

$$=-\int_{t_0}^{t_1}([\partial f/\partial s][\partial s/\partial c_1]+[\partial f/\partial c])\,\mathrm{d}t \tag{70}$$

$$=-\int_{t_0}^{t_1}[\partial s_0/\partial s][\partial f/\partial c]\,\mathrm{d}t \tag{71}$$

hold for the inverse parameter partials of Eq. (54), with respect to c_1.

In the integral equations above, some of the integrands contain notation based on the fact that s_1, c_1, t_1 in Eqs. (51-53) can be regarded as new initial values of s, c, t, which replace s_0, c_0, t_0. Thus

$$s=\sigma(s_1,c_1,t_1,t) \tag{72}$$

corresponds to Eq. (8), and its partials

$$\partial s/\partial s_1 \equiv \partial\sigma(s_1,c_1,t_1,t)/\partial s_1 \tag{73}$$

$$\partial s/\partial c_1 \equiv \partial\sigma(s_1,c_1,t_1,t)/\partial c_1 \tag{74}$$

correspond to Eqs. (15) and (16). Also,

$$s_1=\sigma(s,c,t,t_1) \tag{75}$$

corresponds to Eq. (9), and its partials

$$\partial s_1/\partial s \equiv \partial\sigma(s,c,t,t_1)/\partial s \tag{76}$$

$$\partial s_1/\partial c \equiv \partial\sigma(s,c,t,t_1)/\partial c \tag{77}$$

correspond to Eqs. (19) and (20).

Suppose that the results in Eqs. (51-71) have all been determined for the particular value t_1 of t. Further, suppose that Eqs. (72-77) have been determined for some other value of t. Then the chain rules

$$\partial s/\partial s_0 = [\partial s/\partial s_1][\partial s_1/\partial s_0] \tag{78}$$

$$\partial s/\partial c_0 = [\partial s/\partial s_1][\partial s_1/\partial c_0] + [\partial s/\partial c_1] \tag{79}$$

and

$$\partial s_0/\partial s = [\partial s_0/\partial s_1][\partial s_1/\partial s] \tag{80}$$

$$\partial s_0/\partial c = [\partial s_0/\partial s_1][\partial s_1/\partial c] + [\partial s_0/\partial c_1] \tag{81}$$

give the partials Eqs. (15), (16) and Eqs. (19), (20) above. The results of Eqs. (72-81) are particularly valuable in stepwise solutions where t_1 is treated as a new initial value that replaces t_0, but results for the original initial value t_0 are to be determined.

Equation (63)[2] for the parameter partials $\partial s_1/\partial c_0$ is a quadrature in the sense that the result $\partial s_1/\partial c_0$ is not the particular value of a quantity that occurs in the integrand. Thus, the integrand of Eq. (63) is a function of t alone once $\partial s_1/\partial s$ and $\partial f/\partial c$ have been determined as functions of t. Then numerical quadrature formulas can be used to evaluate the quadrature formula Eq. (63) for the parameter partials $\partial s_1/\partial c_0$. Similar remarks hold for the quadrature formula Eq. (71) for the inverse parameter partials $\partial s_0/\partial c_1$.

Parts of the results given above can be selected and combined in many different ways to solve a particular variational problem. The particular selection largely depends on personal preferences and the particular problem at hand. All the results given above have been displayed to provide the greatest generality for such applications.

The next section describes the paticular application to the ground tracking system for Space Shuttle vehicle and payload orbits. The method is being incorporated into the ground tracking system at the Johnson Space Center because it is very efficient in terms of computation and storage requirements for large digital computers. But for those not particularly interested in this application, it should be useful as an example to clarify how the general results given above can be applied to other important, practical variational problems.

Example and Algorithm

The differential equations of motion (1) for a Space Shuttle vehicle or payload orbit have the form

$$\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} p \\ v \end{bmatrix}=\begin{bmatrix} v \\ -\mu p/r^3 \end{bmatrix}+\begin{bmatrix} 0 \\ g(s,c,t) \end{bmatrix}\equiv f(s,c,t) \tag{82}$$

where

$$t=t_0 \Rightarrow p=p_0 \quad \text{and} \quad v=v_0 \tag{83}$$

are the initial conditions. Here the independent variable t is the ephemeris time. The first three components of the six-dimensional state vector s are the respective three components of the vector p of inertial Cartesian position coordinates. The second three components of the vector s are the respective three components of the inertial velocity vector $v\equiv \mathrm{d}p/\mathrm{d}t$.

The gravitational constant for the Earth is $\mu=G\times M$ where G is the Newtonian constant of universal gravitation, and M is the total mass of the Earth. Here

$$r=(p^Tp)^{1/2}>0 \tag{84}$$

is the magnitude of the position vector p. The two-body acceleration is accordingly $-\mu p/r^3$. The perturbing acceleration g is in general a function of s and t, as well as a vector c of parameters that do not change with time along a trajectory, but can be varied to fit a trajectory to given radar measurements of a spacecraft. The function g can also depend on other parameters, but these are not considered when they are not varied for such a fit.

The determination of the initial values p_0 and v_0 at a given initial time t_0 to fit given radar measurements is an important function that has been performed by the ground tracking system at the Johnson Space Center for many years. This in turn has required the determination of the 6×6 state transition matrix $\partial s/\partial s_0$ of partial derivatives for many different values of the time t. A closed-form matrix function

$$\partial s/\partial s_0=E(\mu,s,s_0,t-t_0) \tag{85}$$

originated by the present author[1] has been used to do this for many years. The matrix function E is exact for unperturbed, two-body motion $g\equiv 0$, but is used as an approximation for the perturbed motion $g\not\equiv 0$.

In this example Eq. (82), the three matrices Eqs. (11-13) are

$$\frac{\partial f}{\partial s}=\begin{bmatrix} 0 & I \\ (3pp^T/r^2-I)\mu/r^3 & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ \partial g(s,c,t)/\partial p & \partial g(s,c,t)/\partial v \end{bmatrix} \tag{86}$$

$$\frac{\partial f}{\partial c}=\begin{bmatrix} 0 \\ \partial g(s,c,t)/\partial c \end{bmatrix} \tag{87}$$

$$\frac{\partial f}{\partial t}=\begin{bmatrix} 0 \\ \partial g(s,c,t)/\partial t \end{bmatrix} \tag{88}$$

respectively. The matrices Eqs. (15), (16) and Eqs. (19), (20) are

$$\frac{\partial s}{\partial s_0}=\begin{bmatrix} \partial p/\partial p_0 & \partial p/\partial v_0 \\ \partial v/\partial p_0 & \partial v/\partial v_0 \end{bmatrix} \tag{89}$$

$$\frac{\partial s}{\partial c_0}=\begin{bmatrix} \partial p/\partial c_0 \\ \partial v/\partial c_0 \end{bmatrix} \tag{90}$$

$$\frac{\partial s_0}{\partial s} = \begin{bmatrix} \partial p_0/\partial p & \partial p_0/\partial v \\ \partial v_0/\partial p & \partial v_0/\partial v \end{bmatrix} \tag{91}$$

$$\frac{\partial s_0}{\partial c} = \begin{bmatrix} \partial p_0/\partial c \\ \partial v_0/\partial c \end{bmatrix} \tag{92}$$

respectively. The basic problem considered in this example is a method of computing the parameter partial derivatives $\partial s/\partial c_0$ in Eq. (90), along with the state partials $\partial s/\partial s_0$ in Eq. (89). The method is an extension and modification of previous methods for determining the solution s of Eq. (82) with the state transition matrix $\partial s/\partial s_0$ in Eq. (89).

The algorithm for determining the parameter partials $\partial s/\partial c_0$, as well as the state partials $\partial s/\partial s_0$ with the solution s, is organized as follows. Let t_0 be taken as the initial value for t_1. Then,

$$t_1 = t_0 \tag{93}$$

$$s_1 = s_0 \tag{94}$$

$$\partial f_1/\partial c_1 = \partial f(s_1,c,t_1)/\partial c \tag{95}$$

$$\partial s_1/\partial s_0 = I \tag{96}$$

$$\partial s_1/\partial c_0 = 0 \tag{97}$$

give initial values for $t_1 = t_0$. Assume integration in the positive direction $t > t_0$, for simplicity, since integration in the negative direction $t < t_0$ is similar. It is possible for the value Eq. (95) of $\partial f/\partial c$ in Eq. (12) to be discontinuous at $t = t_1 = t_0$. If so, the value of $\partial f/\partial c$ in Eq. (12) for $t \rightarrow t_1 < t$ must be selected for $\partial f_1/\partial c_1$ in Eq. (95).

Let $t_2 > t_1$ be a later time for which

$$s_2 = s_1 + \int_{t_1}^{t_2} f(s,c,t)\,\mathrm{d}t \tag{98}$$

corresponding to Eq. (52), is to be determined. The time t_2 can be chosen in different ways, and might be the time a particular quantity is measured, the end of a current step of numerical integration, or a time for which f or $\partial f/\partial c$ have a discontinuity. The time interval $(t_2 - t_1)$ is normally a small time interval, such as that of a numerical integration step. In the ground tracking system for Space Shuttle orbits, the value of s_2 at t_2 in Eq. (98) is calculated by a sophisticated multistep predictor-corrector, numerical integration scheme, and an associated method of interpolation.

Then the state partials

$$\partial s_2/\partial s_1 = I + \int_{t_1}^{t_2} [\partial f/\partial s]\,[\partial s/\partial s_1]\,\mathrm{d}t \tag{99}$$

$$\doteq E(\mu, s_2, s_1, t_2 - t_1) \tag{100}$$

follow where Eq. (99) corresponds to Eq. (58), and Eq. (100) corresponds to Eq. (85). The closed-form expression Eq. (100) for two-body motion approximates $\partial s_2/\partial s_1$ accurately since $(t_2 - t_1)$ is small.

The evaluation

$$\partial f_2/\partial c_2 = \partial f(s_2,c,t_2)/\partial c \tag{101}$$

of $\partial f/\partial c$ in Eq. (12) for $t = t_2$ is also required. If $\partial f/\partial c$ is discontinuous at $t = t_2$, then Eq. (101) must be the value for $t \rightarrow t_2 > t$. Then the parameter partials

$$\partial s_2/\partial c_1 = \int_{t_1}^{t_2} [\partial s_2/\partial s]\,[\partial f/\partial c]\,\mathrm{d}t \tag{102}$$

$$\doteq ([\partial s_2/\partial s_1]\,[\partial f_1/\partial c_1] + [\partial f_2/\partial c_2])\,(t_2 - t_1)/2 \tag{103}$$

follow where Eq. (102) corresponds to the quadrature formula Eq. (63) above. The approximation Eq. (103) is merely the trapezoidal rule for numerical quadrature. It is simply the average of the two values of the integrand in Eq. (102) at t_1 and t_2 multiplied by the time interval $(t_2 - t_1)$. This time interval is always chosen to be sufficiently small so that the trapezoidal rule Eq. (103) gives a good approximation for the parameter partials $\partial s_2/\partial c_1$. The chain rules

$$\partial s_2/\partial s_0 = [\partial s_2/\partial s_1]\,[\partial s_1/\partial s_0] \tag{104}$$

$$\partial s_2/\partial c_0 = [\partial s_2/\partial s_1]\,[\partial s_1/\partial c_0] + [\partial s_2/\partial c_1] \tag{105}$$

corresponding to Eqs. (78) and (79) are then applied. The updates

$$t_1 = t_2 \tag{106}$$

$$s_1 = s_2 \tag{107}$$

$$\partial s_1/\partial s_0 = \partial s_2/\partial s_0 \tag{108}$$

$$\partial s_1/\partial c_0 = \partial s_2/\partial c_0 \tag{109}$$

then prepare for an iteration of the entire procedure. If $\partial f/\partial c$ in Eq. (12) is continuous at the updated value of t_1 in Eq. (106), then

$$\partial f_1/\partial c_1 = \partial f_2/\partial c_2 \tag{110}$$

can also be updated. But if $\partial f/\partial c$ in Eq. (12) is discontinuous at the new value Eq. (106) of t_1, then

$$\partial f_1/\partial c_1 = \partial f(s_1,c,t_1)/\partial c \tag{111}$$

must be evaluated to determine $\partial f/\partial c$ in Eq. (12) for $t \rightarrow t_1 < t$. A new value $t_2 > t_1$ for t_2 is then chosen, and the steps described in Eqs. (98-111) above are repeated. In this way, new values of s_1, $\partial s_1/\partial s_0$, and $\partial s_1/\partial c_0$ ar generated for each new value of t_1. The iterations are continued until $t_1 > t_0$ is as large as desired. The entire procedure is very efficient in terms of computation time and storage requirements for a large digital computer. Therefore, the approach has been incorporated into the ground tracking system at the Johnson Space Center to help control the flight of Space Shuttle vehicles and their payloads in orbit.

Conclusion

The important application discussed in the previous section above is only one example of the usefulness of the quadrature formula for parameter partials and the other general results discussed in the first three sections. The basic advantage of the quadrature formula is that numerical quadrature formulas can be used to compute parameter partials after the solution and state partials have been determined. This computation of parameter partials with state partials for the solution of a system of simultaneous ordinary differential equations is necessary in many important, practical variational problems.

But, the selection from the general results above to construct an efficient algorithm depends on personal preferences, and the particular problem considered.

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